Algebra is Half the Battle: Verifying Presentations for Graded Unipotent Chevalley Groups

ITP 2025

Eric Wang Arohee Bhoja Cayden Codel Noah Singer

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- Automation in need of automation `lin_id_inv_thms`

Warning!

I know about the math we formalized, but not much about the surrounding research context

What we proved

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We proved in Lean that three specific groups can be defined using a canonically smaller set of equations than previously known.

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Our proof strategy was to show that each full relation could be derived from the weak relations. We derived each relation by solving one or more group rewriting problems.

What we proved

```
theorem full_relations_implied_by_weak_relations :
  ∀ r ∈ (fullB3LargeGraded F).allRelations,
    (weakB3LargeGraded F).project r = 1 := by
```

Why we proved it

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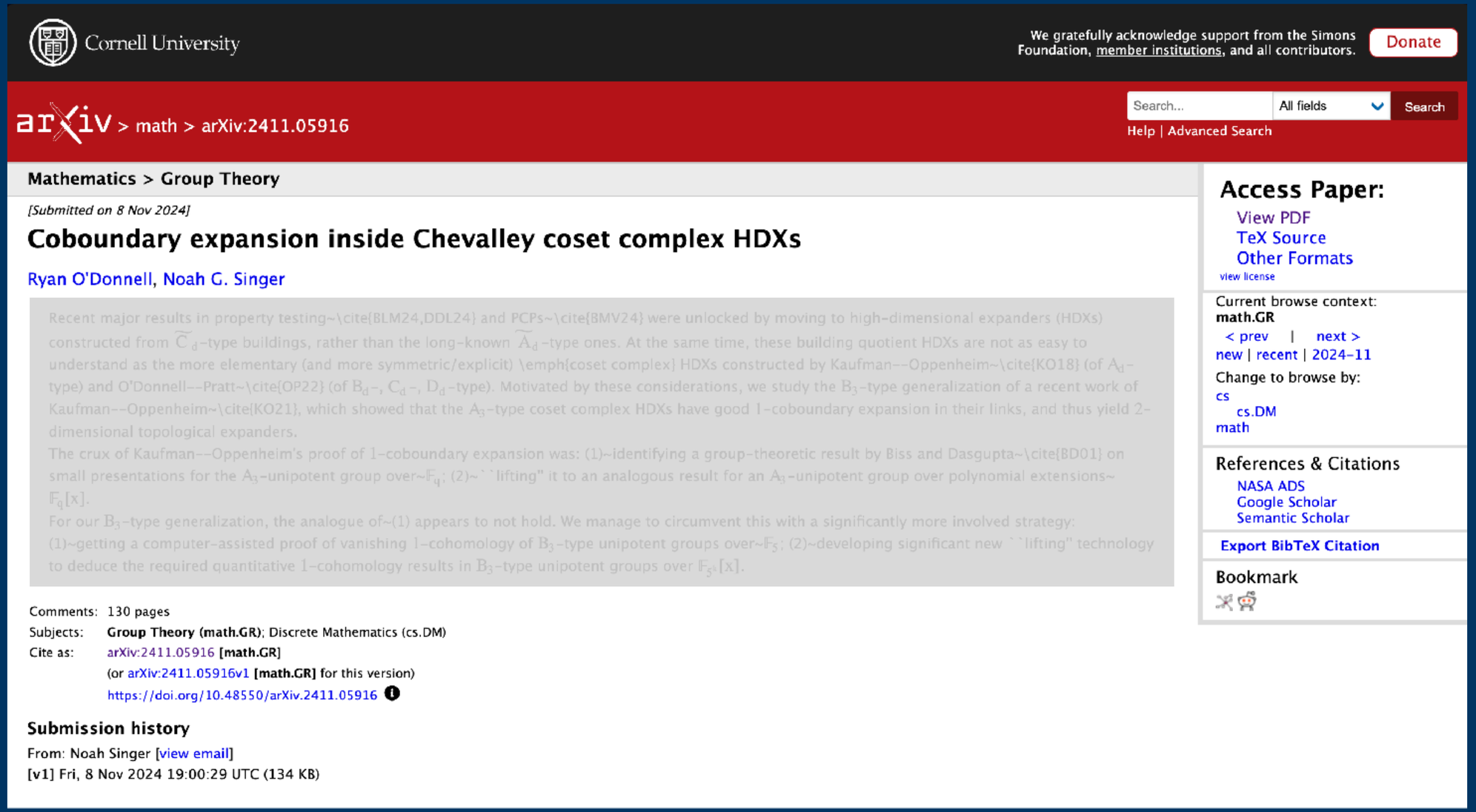

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Mathematics > Group Theory

[Submitted on 8 Nov 2024]

Coboundary expansion inside Chevalley coset complex HDXs

Ryan O'Donnell, Noah G. Singer

Recent major results in property testing~\cite{BLM24,DDL24} and PCPs~\cite{BMV24} were unlocked by moving to high-dimensional expanders (HDXs) constructed from $\widetilde{C}_d$-type buildings, rather than the long-known $\widetilde{A}_d$-type ones. At the same time, these building quotient HDXs are not as easy to understand as the more elementary (and more symmetric/explicit) \emph{coset complex} HDXs constructed by Kaufman--Oppenheim~\cite{KO18} (of A_d-type) and O'Donnell--Pratt~\cite{OP22} (of B_d-, C_d-, D_d-type). Motivated by these considerations, we study the B_3-type generalization of a recent work of Kaufman--Oppenheim~\cite{KO21}, which showed that the A_3-type coset complex HDXs have good 1-coboundary expansion in their links, and thus yield 2-dimensional topological expanders.

The crux of Kaufman--Oppenheim's proof of 1-coboundary expansion was: (1)~identifying a group-theoretic result by Biss and Dasgupta~\cite{BD01} on small presentations for the A_3-unipotent group over~$\mathbb{F}_q$; (2)~``lifting'' it to an analogous result for an A_3-unipotent group over polynomial extensions~$\mathbb{F}_q[x]$.

For our B_3-type generalization, the analogue of~(1) appears to not hold. We manage to circumvent this with a significantly more involved strategy: (1)~getting a computer-assisted proof of vanishing 1-cohomology of B_3-type unipotent groups over~$\mathbb{F}_5$; (2)~developing significant new ``lifting'' technology to deduce the required quantitative 1-cohomology results in B_3-type unipotent groups over $\mathbb{F}_{5^s}[x]$.

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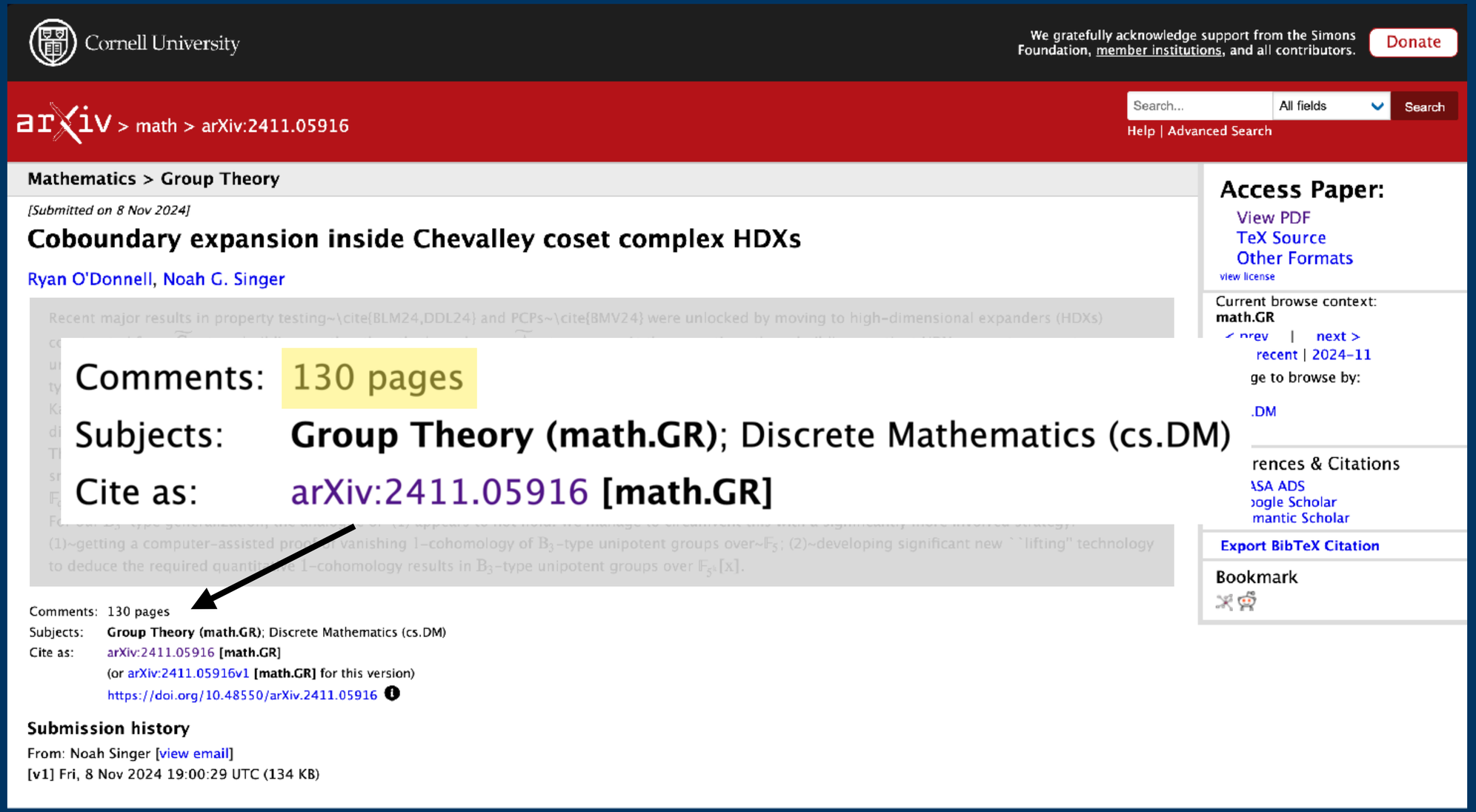
Why we proved it



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Why we proved it

- To verify a result involving hundreds of calculations
  - Constants, negative signs, delicate computations
  - Noah: “What if I made a mistake somewhere?”
- To lay the groundwork for similar verifications
  - Lots of future work left!

Why work on this problem?

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(Specifically, topological expanders)

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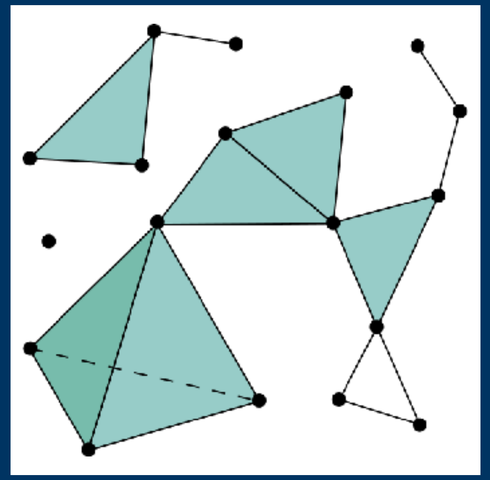
Simplicial complexes



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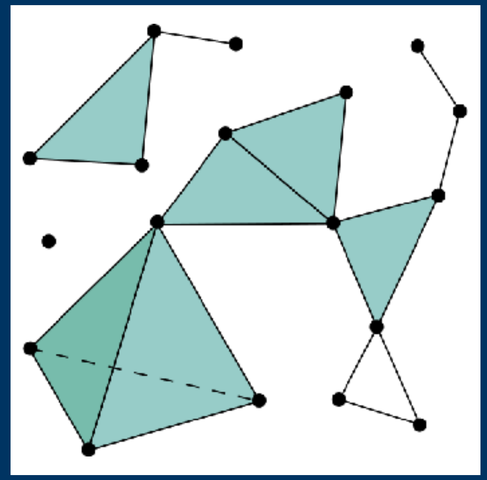
Quotients of Bruhat-Tits buildings



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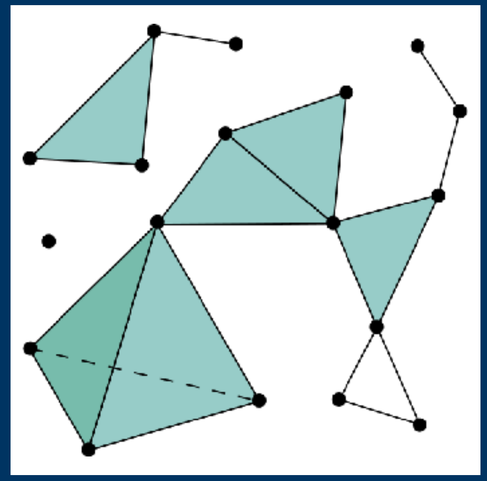

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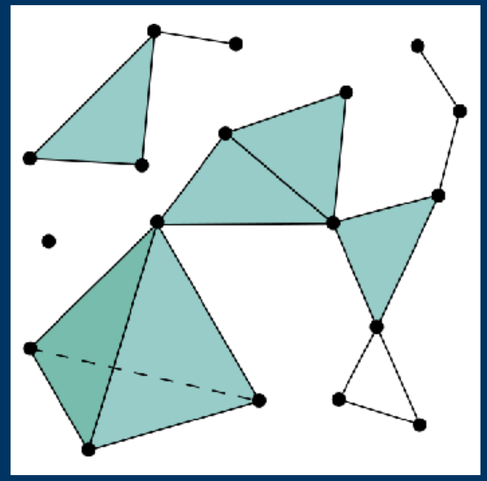
Chevalley groups defined by a good set of Steinberg relations



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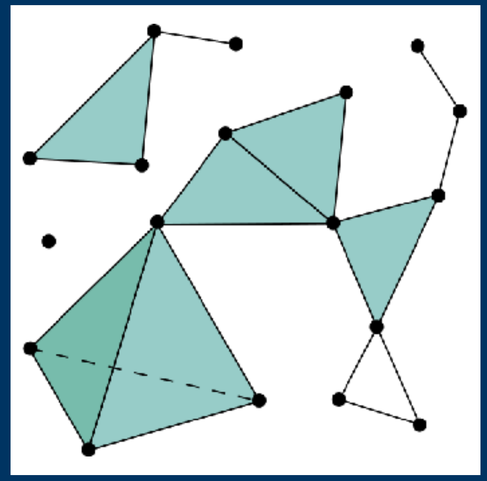
We formalized this part



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  - Useful for (quantum) error correction, local property testing, and higher-dimensional geometry

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- To construct higher-dimensional expanders
  - Useful for (quantum) error correction, local property testing, and higher-dimensional geometry
- To do basic research in group theory

Chevalley groups

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- Similar to Lie groups

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- Defined on square matrices containing field elements

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- Defined on square matrices containing field elements
- Express nice symmetries and geometric properties

Chevalley groups

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$$\{\zeta,\ t\} \qquad\qquad \{\zeta,\ i,\ t\}$$

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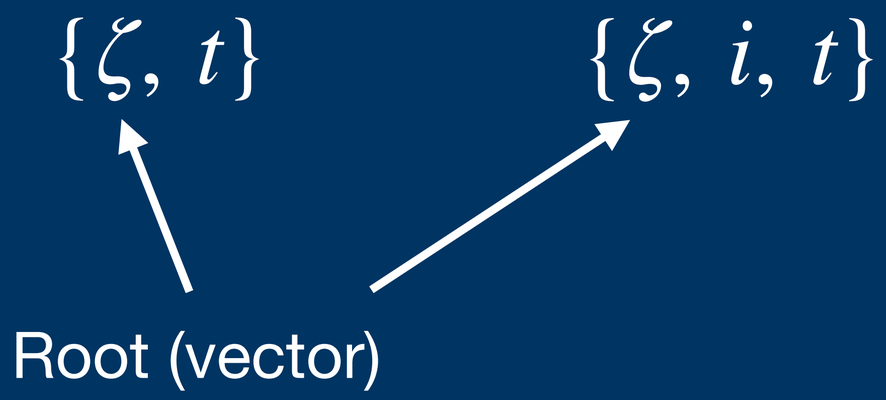

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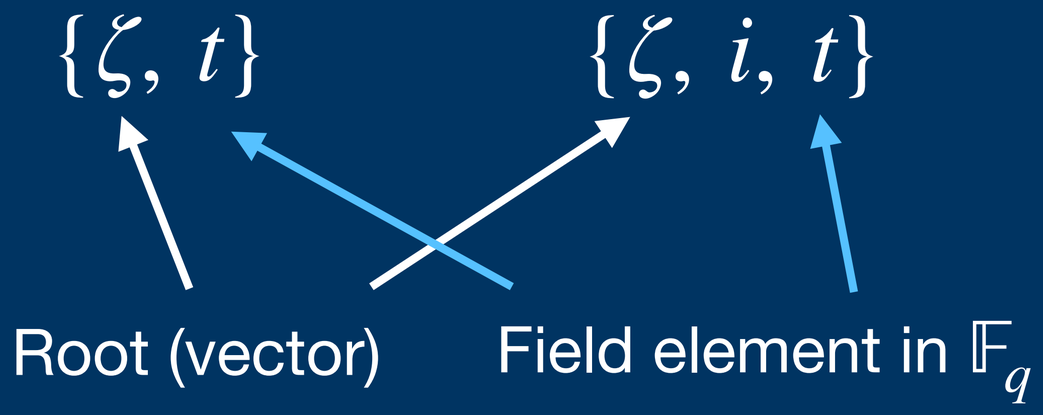

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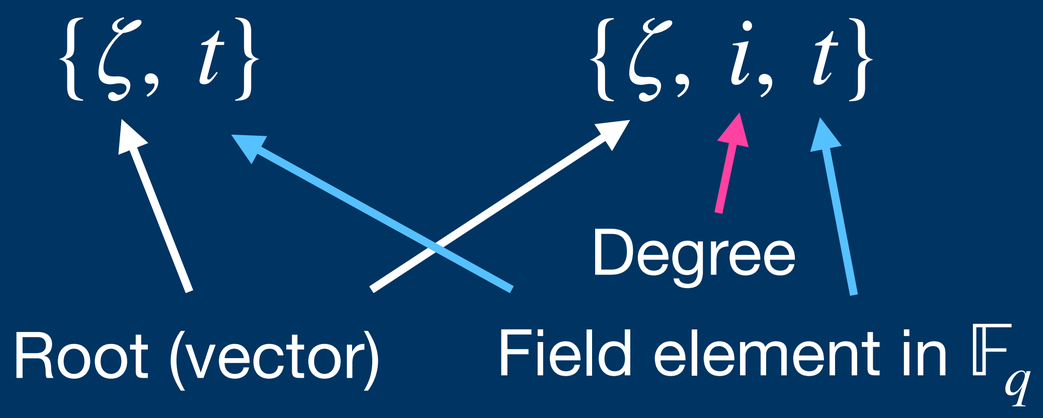

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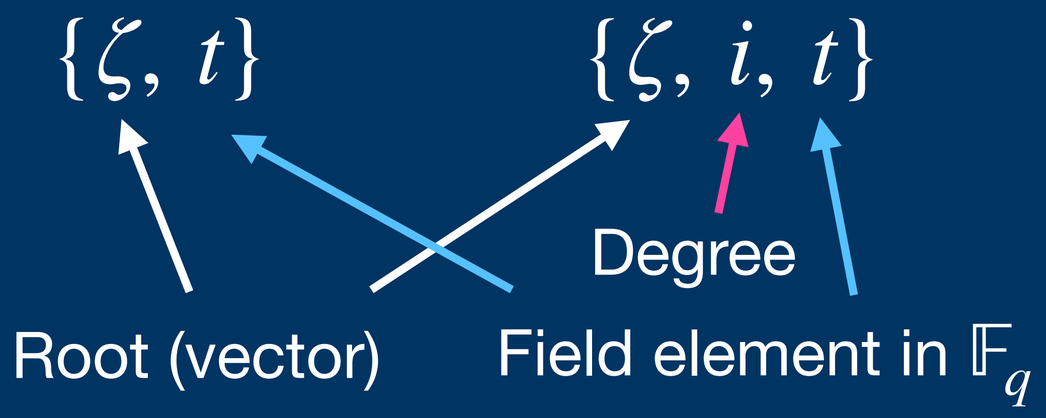


```
structure GradedChevalleyGenerator
    (Φ : Type TΦ) [PositiveRootSystem Φ]
    (R : Type TR) [Ring R]
where
  ζ : Φ -- root
  i : ℕ -- degree
  hi : i ≤ height ζ
  t : R -- field coefficient
```

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  - Commutator: For any pair of roots ζ and η, the commutator is a product of generators of the following form, for Chevalley constants a, b, and $C_{\zeta,\eta}^{a,b} \in [-3,3]$:

$$[\{\zeta, i, t\}, \{\eta, j, u\}] = \prod \{a\zeta + b\eta, ai + bj, C_{\zeta,\eta}^{a,b}\, t^a u^b\}$$

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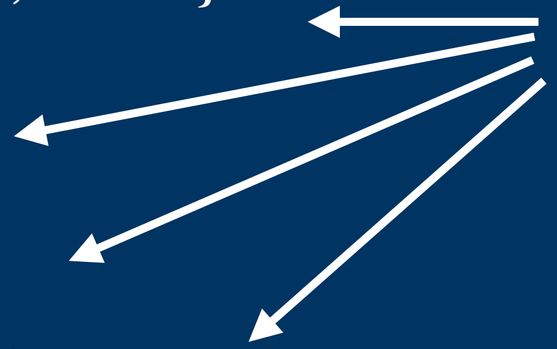

These are (some of the) Steinberg relations

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Equivalently, G is the free group on S modulo the normal closure of R in S

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An example calculation

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Assume the weak Steinberg relations. Then we want to show the full relation

$$\{\beta + \psi + \omega, i, t\}\{\beta + \psi + \omega, i, u\} = \{\beta + \psi + \omega, i, t + u\}$$

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&= \{\beta, i_1, t\}\{\beta, i_1, u\}\{\psi+\omega, i_2, 1\}\{\beta, i_1, -u\}\{\psi+\omega, i_2, -1\}\{\psi+\omega, i_2, 1\}\{\beta, i_1, -t\}\{\psi+\omega, i_2, -1\} \\
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$$\begin{aligned}
&\{\beta+\psi+\omega, i, t\}\{\beta+\psi+\omega, i, u\} \\
&\quad = \{\beta, i_1, t\}\{\psi+\omega, i_2, 1\}\{\beta, i_1, -t\}\{\psi+\omega, i_2, -1\}\{\beta+\omega+\psi, i, u\} \\
&\quad = \{\beta, i_1, t\}\{\beta+\omega+\psi, i, u\}\{\psi+\omega, i_2, 1\}\{\beta, i_1, -t\}\{\psi+\omega, i_2, -1\} \\
&= \{\beta, i_1, t\}\{\beta, i_1, u\}\{\psi+\omega, i_2, 1\}\{\beta, i_1, -u\}\{\psi+\omega, i_2, -1\}\{\psi+\omega, i_2, 1\}\{\beta, i_1, -t\}\{\psi+\omega, i_2, -1\} \\
&\quad = \{\beta, i_1, t+u\}\{\psi+\omega, i_2, 1\}\{\beta, i_1, -u\}\{\psi+\omega, i_2, 0\}\{\beta, i_1, -t\}\{\psi+\omega, i_2, -1\} \\
&\quad = \{\beta, i_1, t+u\}\{\psi+\omega, i_2, 1\}\{\beta, i_1, -(t+u)\}\{\psi+\omega, i_2, -1\} \\
&\quad = \{\beta+\psi+\omega, i_1+i_2, t+u\}
\end{aligned}$$

An example calculation

Assume the weak Steinberg relations. Then we want to show the full relation

$$\{\beta+\psi+\omega,i,t\}\{\beta+\psi+\omega,i,u\}=\{\beta+\psi+\omega,i,t+u\}$$

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&\quad=\{\beta,i_1,t\}\{\beta+\omega+\psi,i,u\}\{\psi+\omega,i_2,1\}\{\beta,i_1,-t\}\{\psi+\omega,i_2,-1\}\\
&=\{\beta,i_1,t\}\{\beta,i_1,u\}\{\psi+\omega,i_2,1\}\{\beta,i_1,-u\}\{\psi+\omega,i_2,-1\}\{\psi+\omega,i_2,1\}\{\beta,i_1,-t\}\{\psi+\omega,i_2,-1\}\\
&\quad=\{\beta,i_1,t+u\}\{\psi+\omega,i_2,1\}\{\beta,i_1,-u\}\{\psi+\omega,i_2,0\}\{\beta,i_1,-t\}\{\psi+\omega,i_2,-1\}\\
&\quad=\{\beta,i_1,t+u\}\{\psi+\omega,i_2,1\}\{\beta,i_1,-(t+u)\}\{\psi+\omega,i_2,-1\}\\
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Proof. Decompose i arbitrarily into $i = i_1 + i_2$. Then …

```
@[simp, chev_simps]
theorem lin_of_βψω : ∀ (i : N) (t u : F),
    {βψω, i, t} * {βψω, i, u} = {βψω, i, t + u} := by
  intro i hi t u
  rcases decompose i with ⟨i₁, i₂⟩    -- i = i₁ + i₂
  grw [ expr_βψω_as_β_ψω_β_ψω,                                  -- Line 1
        expr_βψω_ψω_as_ψω_βψω, expr_βψω_β_as_β_βψω,              -- Line 2
        expr_βψω_ψω_as_ψω_βψω,                                  -- Line 3
        mul_one u, expr_βψω_as_β_ψω_β_ψω,                       -- Line 5
        mul_one (t + u), expr_βψω_as_β_ψω_β_ψω ]                -- Line 6
```

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Proof. Decompose i arbitrarily into $i = i_1 + i_2$. Then …

```
@[simp, chev_simps]
theorem lin_of_βψω : lin_of_root((weakB3SmallGraded F).project, βψω) :=
by
  intro i hi t u
  rcases decompose 1 2 i hi with ⟨i₁, i₂, rfl, hi₁, hi₂⟩
  rw [←mul_one t, expr_βψω_as_β_ψω_β_ψω Fchar hi₁ hi₂]
  grw [←expr_βψω_ψω_as_ψω_βψω Fchar, ←expr_βψω_β_as_β_βψω Fchar,
  ←expr_βψω_ψω_as_ψω_βψω Fchar]
  rw [←mul_one u]
  grw [expr_βψω_as_β_ψω_β_ψω Fchar hi₁ hi₂]
  rw [←mul_one (t + u)]
  grw [expr_βψω_as_β_ψω_β_ψω Fchar hi₁ hi₂]
  ring_nf
```

Our proof strategy

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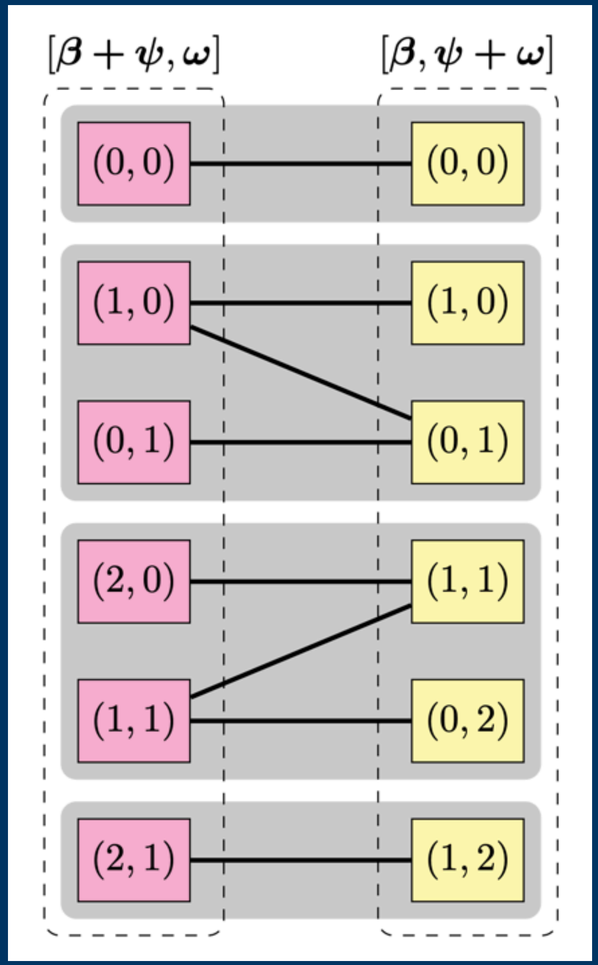

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Sometimes it’s worse:

```
theorem expand_βψω_as_commutator_of_βψ_ω :
  forall_ij_tu 2 1, ⁅⟪βψω, i + j, 2 * t * u⟫ = [⟪βψ, i, t⟫, ⟪ω, j, u⟫]⁆ := by
  intro i j hi hj t u
  match i, j with
  | 0, 0 => rw [expr_βψω_as_comm_of_βψ_ω_00 Fchar]
  | 0, 1 => rw [expr_βψω_as_comm_of_βψ_ω_01 Fchar]
  | 1, 0 => rw [expr_βψω_as_comm_of_βψ_ω_10 Fchar]
  | 1, 1 => rw [expr_βψω_as_comm_of_βψ_ω_11 Fchar]
  | 2, 0 => rw [expr_βψω_as_comm_of_βψ_ω_20 Fchar]
  | 2, 1 => rw [expr_βψω_as_comm_of_βψ_ω_21 Fchar]
```

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Sometimes it’s better (i.e. we get proofs for free):

```
-- height 2 (reflection of height 1)
declare_B3Small_reflected_thm F b3small_valid βψω β ψω const 1 heights 2 1 1 to 1 0 1
declare_B3Small_reflected_thm F b3small_valid βψω β ψω const 1 heights 2 0 2 to 1 1 0
declare_B3Small_reflected_thm F b3small_valid βψω βψ ω const 2 heights 2 2 0 to 1 0 1
declare_B3Small_reflected_thm F b3small_valid βψω βψ ω const 2 heights 2 1 1 to 1 1 0
declare_B3Small_reflected_thm F b3small_valid βψω β ψω const 1 heights 3 1 2 to 0 0 0
declare_B3Small_reflected_thm F b3small_valid βψω βψ ω const 2 heights 3 2 1 to 0 0 0
```

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In this paper, we work with root systems

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inductive B3SmallPosRoot
  | β | ψ | ω | βψ | ψω | β2ψ | βψω
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Some type isomorphisms are better than others

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```
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declare_B3Small_lin_id_inv_thms F ψ
declare_B3Small_lin_id_inv_thms F ω
declare_B3Small_lin_id_inv_thms F βψ
declare_B3Small_lin_id_inv_thms F ψω
declare_B3Small_lin_id_inv_thms F β2ψ

declare_B3Small_trivial_span_of_root_pair_thms F β βψ
declare_B3Small_trivial_span_of_root_pair_thms F β β2ψ
declare_B3Small_trivial_span_of_root_pair_thms F ψ β2ψ
declare_B3Small_trivial_span_of_root_pair_thms F βψ β2ψ
declare_B3Small_trivial_span_of_root_pair_thms F β ω
declare_B3Small_trivial_span_of_root_pair_thms F ψ ψω
declare_B3Small_trivial_span_of_root_pair_thms F ω ψω

declare_B3Small_single_span_of_root_pair_thms F ψ ω ψω 2
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/-! ### Mixed-degree theorem for specific roots -/

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Solution: a macro to declare macros?

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$$\{\beta, i, t\} \cdot \{\beta, i, -t\} \implies \{\beta, i, 0\} \implies 1$$

Other pain points

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Presentation relations vs. equations

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$$[\{\beta, i, t\}, \{\beta + \psi, j, u\}] = 1$$

$$\Leftrightarrow$$

$$\{\beta, i, t\} \cdot \{\beta + \psi, j, u\} \cdot \{\beta, i, -t\} \cdot \{\beta + \psi, j, -u\} = 1$$

$$\Leftrightarrow$$

$$\{\beta, i, t\} \cdot \{\beta + \psi, j, u\} = \{\beta + \psi, j, u\} \cdot \{\beta, i, t\}$$

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Group/field arithmetic often needed massaging to discharge with automation

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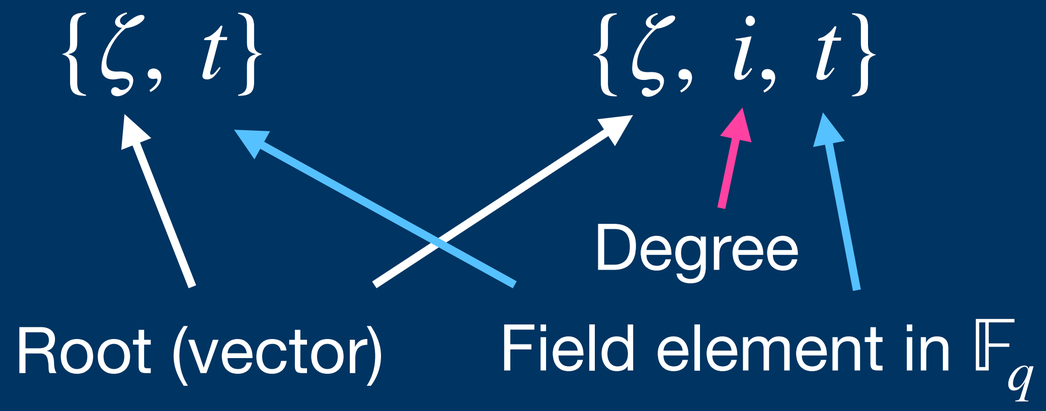
```
have aux₁ : 2 * (u / (2 * t)) = u / t := by ring_nf; field_simp; group
have aux₂ : u / (2 * t) * (u / (2 * t)) = (u * u) / (4 * (t * t)) := by
  ring_nf; simp only [inv_pow, mul_eq_mul_left_iff, inv_injective, mul_eq_zero, ne_eq,
    OfNat.ofNat_ne_zero, not_false_eq_true, pow_eq_zero_iff, inv_eq_zero];
  left
  rw [pow_two, mul_two, two_add_two_eq_four]
```

Thank you for your attention

`https://github.com/singerng/steinberg-formalization`



$$\alpha \mid \beta \mid \alpha + \beta$$

theorem `lin_of_`α

$$a * (b * c) = (a * b) * c$$

`lin_id_inv_thms`